ON THE GONALITY SEQUENCE OF AN ALGEBRAIC CURVE

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ABSTRACT. For any smooth irreducible projective curve X, the gonality sequence $\{d_r \mid r \in \mathbb{N}\}$ is a strictly increasing sequence of positive integer invariants of X. In most known cases d_{r+1} is not much bigger than d_r. In our terminology this means the numbers d_r satisfy the slope inequality. It is the aim of this paper to study cases when this is not true. We give examples for this of extremal curves in $\mathbb{P}^r$, for curves on a general K3-surface in $\mathbb{P}^r$ and for complete intersections in $\mathbb{P}^3$.

1. INTRODUCTION

Let X be a complex smooth irreducible projective curve of genus $g \geq 4$. If we understand by the degree of a rational map $X \to \mathbb{P}^r$ the product of the usual degree of the map onto its image times the degree of the closure of its image in $\mathbb{P}^r$, then, for any positive integer r, the invariant $d_r = d_r(X)$ of X is defined to be the smallest number d such that X admits a nondegenerate rational map of degree d into $\mathbb{P}^r$. The invariant d_1 is the usual gonality of X. Therefore the sequence $d_1, d_2, d_3, \dots$ is called the *gonality sequence* of X.

For any curve and any $r \geq g$, the numbers d_r are known by Riemann-Roch. Hence there are only finitely many interesting numbers in a gonality sequence. By Brill-Noether theory the whole sequence is known for general curves of genus g. Moreover it is known for smooth plane, hyperelliptic, trigonal, general tetragonal, general pentagonal and bielliptic curves (for the references see [10] and Proposition 4.2 below). Apart from that the knowledge on the numbers d_r seems to be scarce. It is the aim of this note to give a first systematic investigation of these numbers.

Clearly the gonality sequence plays an essential rôle in the theory of special curves. Recently it turned out that it also is important for Brill-Noether theory of vector bundles on X. In fact, most of the proofs in [10] use how some of the numbers d_r are related to each other. In

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"most" cases they satisy the inequality

$$\frac{d_r}{r} \geq \frac{d_{r+1}}{r+1}.$$

It is true for all r and all the examples of curves mentioned above, apart from a few numbers r for smooth plane curves (see Proposition 4.3). We call it the *slope inequality* (for the gonality sequence) and it was the main motivation for us to find more counter-examples to its validity for all r.

In Corollary 4.6 we show that the slope inequality is violated for any $r \geq 2$ and any extremal curve of degree $d = 3r - 1$ in $\mathbb{P}^r$. More generally, we prove that the curves of the following three classes do not satisfy all slope inequalities:

(1) extremal curves of degree $\geq 3r-1$ in $\mathbb{P}^r$ ($r \geq 2$; Theorem 4.13),
(2) smooth curves of degree $\geq 8(r-1)$ on a general K3-surface of degree $2(r-1)$ in $\mathbb{P}^r$ ($r \geq 3$; Theorem 5.3),
(3) smooth complete intersections of surfaces of degree s and p with $2 \leq s \leq p$ and $p \geq 4$ in $\mathbb{P}^3$ (Theorem 6.1).

Section 2 contains some preliminaries, mainly on extremal curves. In Section 3 we recall some results on the gonality sequence and prove some new ones. Section 4 contains the proofs of the above mentioned results concerning extremal curves, as well as some consequences and examples. Finally in Section 5 respectively 6 we prove the results on curves on a general K3-surface respectively on complete intersection curves in $\mathbb{P}^3$. Moreover, the case of complete intersection curves in $\mathbb{P}^3$ is generalized to Halphen curves in $\mathbb{P}^3$.

2. Preliminaries

Let X denote an irreducible smooth projective curve of genus $g \geq 1$ over the field of complex numbers. We recall Castelnuovo's genus bound, following [2], [3] or [6]: Let g^r_d be a simple linear series on X such that $d > r > 1$, and let $\chi(d,r)$ denote the unique positive integer satisfying

$$\frac{d-1}{r-1} - 1 \leq \chi(d,r) < \frac{d-1}{r-1}.$$

Then we have for the genus g of X,

$$g \leq \pi(d,r) := \chi(d,r)\left(\frac{\chi(d,r)-1}{2}(r-1) + \epsilon\right) \tag{2.1}$$

where we write $d - 1 = \chi(d,r)(r-1) + \epsilon$ with $1 \leq \epsilon \leq r-1$. If the upper bound is attained, the g^r_d is complete and very ample and the curve $X \subset \mathbb{P}^r$ is called an *extremal curve*.

If $d < 2r$, the g^r_d is non-special and we obtain $\pi(d,r) = d - r$. If $2r \leq d < 3r - 1$, one computes $\pi(d,r) = 2d - 3r + 1$. In particular we have $\pi(d,r) < d$ for $d < 3r - 1$.

In view of Theorem 4.13 below we are interested in the set of genera of extremal curves of degree $d \geq 3r - 1$ in $\mathbb{P}^r$. So let $d \geq 3r - 1$. By definition of $\chi := \chi(d,r)$ this is equivalent to $\chi \geq 3$,

According to (2.1) χ divides $\pi(d,r)$ if χ is odd, and $\frac{\chi}{2} \geq 2$ divides $\pi(d,r)$ if χ is even. For $r \geq 3$ we have

$$\frac{\chi - 1}{2}(r-1) + \epsilon \geq \frac{\chi - 1}{2} \cdot 2 + 1 = \chi \geq 3,$$

and for $r = 2$ we have

$$3 \leq \chi = d - 2 \quad \text{and} \quad \frac{\chi - 1}{2}(r-1) + \epsilon = \frac{d-1}{2} \geq 2.$$

In particular, $\pi(d,r)$ cannot be a prime number, and $\pi(d,r) \neq 1, 4, 8$. We claim that $\pi(d,r) \neq 14$, too. This is clear for $r = 2$ and for $r \geq 3$ we have $\pi(d,r) \geq \chi^2$. So χ would have to be 7 for odd k and $\chi \geq 4$ for even k.

Now consider the set

$$C := \{\pi(d,r) \mid r \geq 2 \text{ and } \chi(d,r) \geq 3\}$$

of genera of extremal curves of degree $d \geq 3r - 1$ in $\mathbb{P}^r$.

Proposition 2.1. *Let* $3 \leq g \in \mathbb{Z}$, $g \neq 4, 8, 14$. *Then* $g \notin C$ *implies that* g *is odd with at most two different prime factors.*

Proof. First we show that any even positive integer $g \neq 2, 4, 8, 14$ is in C.

In fact, for $d = 3r - 1$, $r \geq 2$ we have $\pi(d,r) = 3r = 6, 9, 12, \cdots$, for $d = 4r - 2$, $r \geq 2$ we obtain $\pi(d,r) = 6r - 2 = 10, 16, 22, \cdots$ and for $d = 4r - 1$, $r \geq 3$ we get $\pi(d,r) = 6r + 2 = 20, 26, 32, \cdots$.

Next we prove the following assertion: Suppose $g \geq 3$ is an odd integer which is not a prime number, and let p be the smallest prime divisor of g. If $g \geq \frac{1}{4}p^2(p+1)$, then $g \in C$.

Note that the assertion implies that any positive odd integer g with at least 3 (not necessarily distinct) prime divisors lies in C.

To prove the assertion, let $g = p \cdot \alpha$ with $p \leq \alpha \in \mathbb{Z}$. For an integer $r \geq 2$ we can write

$$\frac{p-1}{2}(r-1) + 1 \leq \alpha \leq \frac{p-1}{2}(r-1) + (r-1) = \frac{p+1}{2}(r-1)$$

if and only if

$$\frac{2\alpha}{p+1} \leq r - 1 \leq \frac{2\alpha - 2}{p-1}.$$

We can find such an integer r if and only if

$$\lceil \frac{2\alpha}{p+1} \rceil \leq \lfloor \frac{2\alpha - 2}{p-1} \rfloor.$$

We have $\lfloor \frac{2\alpha-2}{p-1} \rfloor = \frac{2\alpha-2-i}{p-1} \geq \frac{2\alpha-2-(p-2)}{p-1}$ for some i, $0 \leq i \leq p-2$. Thus, if

$$\frac{2\alpha - p}{p-1} \geq \frac{2\alpha}{p+1}, \tag{2.2}$$

we have $\lfloor \frac{2\alpha-2}{p-1} \rfloor \geq \frac{2\alpha}{p+1}$ and thus $\lfloor \frac{2\alpha-2}{p-1} \rfloor \geq \lceil \frac{2\alpha}{p+1} \rceil$. So (2.2) implies that we can find an integer r we want, and if we set $\epsilon := \alpha - \frac{p-1}{2}(r-1)$, we have

$$g = p\alpha = p\left(\frac{p-1}{2}(r-1) + \epsilon\right) \quad \text{with} \quad 1 \leq \epsilon \leq r-1.$$

Setting $d := p(r-1) + 1 + \epsilon$, we can find an extremal curve of degree d in $\mathbb{P}^r$ of genus g (with $\chi = p \geq 3$), i.e. $g \in C$.

Obviously (2.2) is equivalent to $4\alpha \geq p(p+1)$. So this proves the assertion.

We have already seen that the assertion implies that an odd positive integer $g \notin C$ can have at most two prime factors. Now if $g = p^2$ (p an odd prime), we may take an extremal space curve of degree $d = 2p+2$. This curve has genus p^2, showing that $p^2 \in C$. □

Remark 2.2. The proof of the assertion shows that if $p, p+2$ are twin primes, then $p(p+2) \notin C$, unless $p \leq 5$. If, however, p and q are odd primes such that $q \geq \frac{p(p+1)}{4}$, then $pq \in C$ by the assertion. □

The paper [6] is somewhat difficult to obtain. So for the convenience of the reader we close this section by recalling two results of it on curves in $\mathbb{P}^r$ of "sufficiently high" genus, which will be needed in the sequel. For the following result see [6, Teorema 2.11].

Theorem 2.3. *Let g^r_d and g^s_m be linear series on X such that g^r_d is base point free and simple,*

$$m \leq kd \quad \text{and} \quad s \geq \binom{k+1}{2}(r-1) + k$$

for some integer k, $1 \leq k < \chi(d,r)$. Then

$$g^s_m = kg^r_d \quad \text{or} \quad g \leq \pi(d,r) - \chi(d,r) + k.$$

The second result is part of [6, Osservatione 2.19].

Proposition 2.4. *Let g^r_d be a base point free and simple linear series on X such that*

$$g \geq \pi(d,r) - \chi(d,r) + 2.$$

Then the g^r_d is complete, $\dim(2g^r_d) = 3r-1$ and X is mapped into a surface of degree $r-1$ in $\mathbb{P}^r$ via the morphism given by the g^r_d.

3. The gonality sequence

Let now $g \geq 4$. For any positive integer r the invariant d_r of X is defined as

$$d_r = d_r(X) := \min\{d \mid X \text{ admits a linear series } g^r_d\}.$$

So d_1 is the gonality of X, d_2 is the minimal degree of a non-degenerate rational map $X \to \mathbb{P}^2$ etc. The sequence $d_1, d_2, d_3, \cdots$ is called the *gonality sequence* of the curve X. The *Clifford index* of X is defined as usual as

$$\gamma := \min\{d - 2r \mid X \text{ admits a } g^r_d \text{ with } r \geq 1 \text{ and } d \leq g-1\}.$$

We say that a linear series g^r_d *contributes to* γ if $r \geq 1$ and $d \leq g-1$. We say that d_r (respectively a g^r_d or a line bundle defining it) *computes* γ if in addition $\gamma = d_r - 2r$ (respectively $\gamma = d - 2r$). For the following lemma see [10, Lemmas 4.2 and 4.3].

Lemma 3.1. (a) $d_r < d_{r+1}$ *for all* r*;*
(b) *if a line bundle* L *computes* d_r*, then* $h^0(L) = r+1$ *and* L *is generated;*
(c) $d_{r+s} \leq d_r + d_s$ *for any* $r, s \geq 1$*; in particular*

$$d_r \leq r \cdot d_1 \leq r \cdot \frac{g+3}{2} \quad \textit{for any} \quad r;$$

(d) *If* $d_r + d_s = d_{r+s}$*, then* $d_n = nd_1$ *for all* $n \leq r+s$*.*

Lemma 3.2. (a) $d_r = r + g$ *for* $r \geq g$*;*
(b) $d_r = r + g - 1$ *for* $g > r > g - d_1$*;*
(c) $d_r \leq g - \left[\frac{g}{r+1}\right] + r$ *and for a general curve we have*

$$d_r = g - \left[\frac{g}{r+1}\right] + r;$$

(d) $d_r \geq \min\{\gamma + 2r, g + r - 1\}$ *for all* r*.*

Proof. Apart from (b) this follows from the Riemann-Roch theorem, the definition of the Clifford index and Brill-Noether theory.

Proof of (b): Assume that $\dim |K_X - g^r_{d_r}| \geq 1$. Then, by definition of d_1, $\deg |K_X - g^r_{d_r}| = 2g - 2 - d_r \geq d_1$. Since $g^r_{d_r}$ contributes to γ, and from [7, Theorem 2.3] we know that $\gamma \geq d_1 - 3$. Consequently,

$$2g - 2 - d_1 \geq d_r \geq 2r + \gamma \geq 2r + d_1 - 3,$$

which implies $r \leq g - d_1$. Hence, for $g > r > g - d_1$ we obtain $\dim |K_X - g^r_{d_r}| \leq 0$. But $g^r_{d_r}$ is certainly a special linear system. So $\dim |K_X - g^r_{d_r}| = 0$. Now Riemann-Roch implies $d_r = r + g - 1$. □

For specific curves, even of "small" genus, it is in general not easy to compute its gonality sequence.

Example 3.3. Let X be a curve of genus 14 with a g^4_{13} computing its Clifford index $\gamma = 5$. By [7, 3.2.2] the curve X has gonality $d_1 = \gamma + 3 = 8$. Such curves exist and the g^4_{13} is the only linear series on it computing γ (see [7, Example 3.2.7]). Hence $d_2 \geq 10$, $d_3 = 12$, $d_4 = 13$ and the g^4_{13} embeds X into $\mathbb{P}^4$. According to [12, Lemma 4] X has trisecant lines in $\mathbb{P}^4$. The projection with center such a line induces a g^2_{10} on X. So $d_2 = 10$. By Serre duality we get $d_5 = 16$ and $d_6 = 18$. Finally, for $r > 6 = g - d_1$ we know d_r by Lemma 3.2 (a) and (b). $\square$

Note that using Serre duality it is easy to calculate the $d_r \geq g$ with $r \leq g - d_1$ provided all d_s with $d_s < g$ are already known.

Let M_g denote the moduli space of smooth projective curves of genus g (≥ 4). Considering the invariants d_r as functions $d_r : M_g \to \mathbb{Z}$, we have

Proposition 3.4. *For any $r \geq 1$ the function $d_r : M_g \to \mathbb{Z}$ is lower semi-continuous, i.e. may become smaller under specialisation.*

Proof. We have to show that for any d the set

$$M_{(g,r,d)} := \{X \in M_g \mid d_r(X) \leq d\}$$

is closed in M_g.

Fix an integer $n \geq 3$ and let M^n_g denote the moduli variety of curves of genus g with level-n structure. There exists a universal curve

$$\pi^n_g : \mathcal{C}^n_g \to M^n_g$$

of genus g with level-n structure (which we omit in the notation, since we don't need it). According to [8], for any integer d the relative Picard scheme $\mathrm{Pic}^d_{\mathcal{C}^n_g/M^n_g}$ is projective over M^n_g. Since it represents a functor, there exists a universal line bundle $\mathcal{L}_d$ on $\mathcal{C}^n_g \times_{M^d_g} \mathrm{Pic}^d_{\mathcal{C}^n_g/M^n_g}$ of degree d. For any geometric point $x \in \mathrm{Pic}^d_{\mathcal{C}^n_g/M^n_g}$ we denote by L_x the corresponding line bundle on the curve $C_{p(x)}$ corresponding to the point $p(x)$, where $p : \mathrm{Pic}^d_{\mathcal{C}^n_g/M^n_g} \to M^n_g$ denotes the structure morphism. Now it is well known that the function

$$h^0 : \left\{ \begin{array}{ccc} \mathrm{Pic}^d_{\mathcal{C}^n_g/M^n_g} & \to & \mathbb{Z} \\ x & \mapsto & h^0(L_x) \end{array} \right.$$

is upper semi-continuous.

Let $C \mapsto C_0$ denote a specialization in M^n_g. Since the map $M^n_g \to M_g$ is finite, it suffices to show that

$$d_r(C_0) \leq d_r(C).$$

Let L be a line bundle on C computing d_r. Since p is a projective morphism, the specialization $C \mapsto C_0$ lifts to a specialization $L \mapsto L_0$. By upper semicontinuity of h^0 we have $h^0(L_0) \geq h^0(L)$, which implies that $d_r(C_0) \leq d_r(C)$. $\square$

4. The slope inequality

As an immediate consequence of Lemma 3.2 (a),(b),(c) we get the following proposition.

Proposition 4.1. (i) $\frac{d_{g-1}}{g-1} = 2 = \frac{d_g}{g}$;
(ii) $\frac{d_r}{r} > \frac{d_{r+1}}{r+1}$ *for any* $r > g - d_1,\ r \neq g-1$;
(iii) $\frac{d_r}{r} > \frac{d_{r+1}}{r+1}$ *for a general curve* X *and any* $r,\ 1 \leq r \neq g-1$.

We call the inequality

$$\frac{d_r}{r} \geq \frac{d_{r+1}}{r+1} \tag{4.1}$$

a *slope inequality* (for the gonality sequence) and we say that X *satisfies the slope inequalities* if (4.1) is valid for all $r \geq 1$. By Proposition 4.1 this is valid for general curves. It is not difficult to see from [10, Remark 5.5] that also hyperelliptic, trigonal, general tetragonal and bielliptic curves satisfy the slope inequalities. The following proposition shows that this is also true for general pentagonal curves.

Proposition 4.2. *For a general pentagonal curve* X *the gonality sequence is given by*

$$d_r = \begin{cases} 5r & r \leq \lfloor \frac{g-3}{5} \rfloor, \\ \lceil \frac{5r+g-3}{2} \rceil & \text{for} \quad \frac{g-3}{5} < r \leq \lfloor \frac{g-1}{5} \rfloor, \\ r+g-1-\lfloor \frac{g-r-1}{4} \rfloor & \text{for} \quad \frac{g-1}{5} < r \leq g-1, \\ r+g & r \geq g. \end{cases}$$

In particular, X *satisfies the slope inequalities.*

Proof. This follows from [14] and, for $r > \frac{g-1}{5}$, Serre duality and Riemann-Roch. The proof of the last assertion is an immediate computation. □

For smooth plane curves we have however,

Proposition 4.3. *Let* X *be a smooth plane curve of degree* $d \geq 5$. *Then the slope inequality holds for all* r *except if*

$$r = \frac{\alpha(\alpha+3)}{2} \quad \textit{with} \quad 1 \leq \alpha \leq d-4,$$

in which case

$$d_r = \alpha d \quad \textit{and} \quad d_{r+1} = (\alpha+1)d - (\alpha+1).$$

So the slope ineqality (4.1) *is not valid for these values of* r.

Proof. Noether's Theorem [6, Theorem 3.14] says that

$$d_r = \begin{cases} \alpha d - \beta & \textit{if} \quad r < g = \frac{(d-1)(d-2)}{2}, \\ r + \frac{(d-1)(d-2)}{2} & \textit{if} \quad r \geq g. \end{cases}$$

where α and β are the uniquely determined integers with $\alpha \geq 1$ and $0 \leq \beta \leq \alpha$ such that

$$r = \frac{\alpha(\alpha+3)}{2} - \beta.$$

In particular we have $d_1 = d-1,\ d_2 = d,\ d_{g-1} = 2g-2,\ d_g = 2g$. Moreover, apart from $r = 1, g-1$ and the exceptional values of the proposition we have $d_{r+1} = d_r + 1$. So for all these values of r the slope inequality holds.

Now let $r = \frac{\alpha(\alpha+3)}{2}$ with $1 \leq \alpha \leq d-4$. Then

$$r < \frac{(d-3)d}{2} = g-1$$

which gives the values of d_r and d_{r+1}. Now $\frac{d_r}{r} < \frac{d_{r+1}}{r+1}$ is equivalent to $\alpha^2 + (4-d)\alpha + (3-\alpha) < 0$ which is true, since $\alpha \leq d-4$. □

Remark 4.4. Note that the biggest r violating the slope inequality in Proposition 4.3 is

$$r = \frac{(d-4)(d-1)}{2} = g - (d-1) = g - d_1.$$

So in this case part (ii) of Proposition 4.1 is best possible.

This is, however, the only case: *If X is not a smooth plane curve, we always have*

$$d_{g-d_1} = 2g - 2 - d_1 = (g-d_1) + g - 2$$

which implies that $\frac{d_r}{r} \geq \frac{d_{r+1}}{r+1}$ for $r = g - d_1$, by Lemma 3.2 (b).

In fact, the dual of a pencil $g^1_{d_1}$ is a series of degree $2g-2-d_1$ and dimension $g-d_1$. Assume that $d_{g-d_1} < 2g-2-d_1$. Then we can find a series of degree $2g-3-d_1$ and dimension $g-d_1$. But its dual is a $g^2_{d_1+1}$ which is very ample, since otherwise the subtraction of two appropriate points of X would give us a $g^1_{d_1-1}$ on X. Hence X is a smooth plane curve of degree d_1+1. □

Now let $X \subset \mathbb{P}^r$ be an extremal curve of degree $3r-1$. We may assume that X is not isomorphic to a smooth plane curve. According to [3, Section III, Corollary 2.6 (iii)], $X \subset \mathbb{P}^r$ is a semicanonical curve of genus genus $g = 3r = d+1$ (see also [1] for these curves).

Proposition 4.5. *With these assumptions we have*

$$d_r = d = g-1 \text{ and } d_{r+1} = g+2 \text{ for any } r \geq 2.$$

An immediate consequence is,

Corollary 4.6. *For any $r \geq 2$ and any extremal curve X of degree $d = 3r-1$ in $\mathbb{P}^r$ we have*

$$\frac{d_r}{r} < \frac{d_{r+1}}{r+1},$$

i.e. (4.1) *is violated.*

Proof. For $r = 2$ this is a special case of Proposition 4.3. The only other case (see [3]) where X may be isomorphic to a smooth plane curve is $r = 5$: a smooth plane septic is, by $2g^2_7$, also an extremal curve of degree 14 in $\mathbb{P}^5$. In this case the assertion is again a special case of Proposition 4.3.

So we may assume $r \geq 3$ and X is not isomorphic to a smooth plane curve. According to [3, Section III, Theorem 2.5]) X lies on a rational normal scroll surface, whose ruling sweeps out a g^1_4 on X. In particular X admits a 4-secant line ℓ. Projection with center ℓ induces a g^{r-2}_{g-5}. Consequently, X has also a g^{r-2}_{g-4} and thus by dualizing a g^{r+1}_{g+2}.

Assume that X admits a g^{r+1}_{g+1}. By dualizing, X has a g^{r-1}_{g-3}. But then, by [2, Lemma 5.1],

$$\dim |g^r_{g-1} + g^{r-1}_{g-3}| \geq r + 2(r-1), \tag{4.2}$$

which contradicts Clifford's Theorem (since $g = 3r$). Hence X does not admit a g^{r+1}_{g+1} which implies $d_{r+1} = g + 2$.

Since X has no g^{r+1}_{g+1}, it does not admit a g^{r+1}_g. Dualizing we see that X has no g^r_{g-2}. This implies $d_r = g - 1$. □

Example 4.7. *Let X be an extremal curve of degree 8 in $\mathbb{P}^3$. Then $g = 9$ and the gonality sequence of X is*

r	1	2	3	4	5	6	7	8	$r \geq 9$
d_r	4	7	8	11	12	14	15	16	$r+9$

In particular, $\frac{d_3}{3} = \frac{8}{3} < \frac{d_4}{4} = \frac{11}{4}$.

Proof. Since X is tetragonal, we have $d_1 = 4$. From (4.2) we conclude that $d_2 = 7$ and by Proposition 4.5, $d_3 = 8$ and $d_4 = 11$. The dual series of a g^1_4 is a g^5_{12}. This implies $d_5 = 12$ using Lemma 3.1 (a). The other assertions follow from Lemma 3.2 (a) and (b). □

Note that for a general tetragonal curve of genus 9 we have $d_2 = 8$ and $d_3 = 10$, whereas the other values of d_r coincide with the those of Example 4.7 (see [10, Remark 4.5 (c)]). On the other hand, for a bielliptic curve of genus 9, the values of d_r are as in Example 4.7, apart form $d_2 = 6$ and $d_4 = 10$.

It is not difficult to see that, apart from smooth plane quintics, all curves of genus $g \leq 8$ satisfy the slope inequalities.

There is a more general principle showing that the curves of Propositions 4.3 and 4.5 do not satisfy all slope inequalities, namely the simple

Lemma 4.8. *Let X be a curve admitting a g^r_d with $d \geq 2r - 1 \geq 3$ such that*
(1) $d_{r-1} = d - 1$ *and*
(2) $2d \leq g + 3r - 2$.

Then $d_r = d$, the g^r_d is complete and very ample and, if $g^{r'}_{d'}$ denotes the Serre-dual linear system $|K_X - g^r_d|$ of the g^r_d, we have $r' \geq r$ and

$$\frac{d_{r'}}{r'} < \frac{d_{r'+1}}{r'+1}.$$

(Note that $d' = 2g - 2 - d$ and $r' = g - 1 - d + r$.)

Proof. Clearly $d_r \leq d$ and since $d_{r-1} = d - 1$, we have $d_r = d$. If the g^r_d were incomplete or not very ample, the curve X would admit a g^{r-1}_{d-2}. Then $d_{r-1} \leq d - 2$, a contradiction.

Since the g^r_d is simple and $d \geq 2r - 1$, it follows from Castelnuovo's count ([2, Lemma 3.2] that $\dim(2g^r_d) \geq 3r - 1$, and assumption (2) implies that

$$\dim(2g^r_d) = 2d - g + 1 + \dim|K_X - 2g^r_d| \leq 3r - 1 + \dim|K_X - 2g^r_d|.$$

Hence $2g^r_d$ is a special linear series, and we obtain

$$\begin{aligned} r' = \dim|K_X - g^r_d| &= \dim|(K_X - 2g^r_d) + g^r_d| \\ &\geq \dim|K_X - 2g^r_d| + \dim(g^r_d) \geq r. \end{aligned}$$

Since $|K_X - g^r_d|$ is a $g^{r'}_{d'}$, we know that $d_{r'} \leq d'$.

Assume that $d_{r'+1} < d' + 3$. Then X admits a complete $g^{r'+1}_{d'+2}$ which by dualization gives a g^{r-1}_{d-2} contradicting $d_{r-1} = d - 1$. Hence we have $d_{r'+1} \geq d' + 3$.

Since, by (2),

$$d' = d - 2r + 2r' < g - 1 - d + r + 2r' = 3r',$$

we clearly have

$$\frac{d'}{r'} < \frac{d'+3}{r'+1}.$$

So the inequalities $d_{r'} \leq d'$ and $d' + 3 \leq d_{r'+1}$ of above complete the proof of the proposition. □

Remark 4.9. (i) In Lemma 4.8 we have $r' > r$ unless $|2g^r_d| = |K_X|$. In fact, otherwise we have by [2, Lemma 5.1],

$$g - 1 = \dim|g^r_d + (K_X - g^r_d)| \geq r + 2r' = 2g - 2 - 2d + 3r$$

contradicting (2).

In particular, $d \leq g-1$ and hence $d' \geq g-1$. Since we know from the proof of Lemma 4.8 that $d_{r'+1} \geq d'+3$, we get $d_{r'+1} \geq (g-1)+3 = g+2$. Hence Lemma 4.8 cannot discover a violation of the slope inequality for numbers $< g$ of the gonality sequence.

(ii) Lemma 4.8 implies that $d \geq 3r-1$. In fact, we noted in Section 2 that $g \leq \pi(d, r) < d$ for $d < 3r-1$. □

Proposition 4.10. *Let g^r_d be a very ample linear series on X with $d \geq 3r - 1$ and assume that*

$$g > \pi(d, r) - \chi(d, r) + 2.$$

Then we have

(1) $d_{r-1} = \min\{(r-1)d_1, d-1\}$, *and*

(2) $2d \leq g + 3r - 2$.

Proof. For $r = 2$ the assertions are obvious. So suppose $r \geq 3$.

(1): Since $d_r \leq d$, clearly $d_{r-1} \leq d - 1$. By [2, Lemma 5.1] we have

$$\dim(g^r_d + g^{r-1}_{d_{r-1}}) \geq r + 2(r-1) = 3r - 2.$$

Assume that we even have $\dim(g^r_d + g^{r-1}_{d_{r-1}}) \geq 3r-1$. Since by assumption $\chi(d,r) \geq 3$ and $g > \pi(d,r) - \chi(d,r) + 2$, it follows from Theorem 2.3 that $g^r_d + g^{r-1}_{d_{r-1}} = 2g^r_d$, which is absurd. Hence we have

$$\dim(g^r_d + g^{r-1}_{d_{r-1}}) = 3r - 2.$$

Assume that $d_{r-1} \leq d - 2$. We have to show that $d_{r-1} = (r-1)d_1$.

If $g^{r-1}_{d_{r-1}}$ is simple, then $g^r_d - g^{r-1}_{d_{r-1}} \neq \emptyset$ by [2, Lemma 5.2]. But since $d_{r-1} \leq d - 2$, this implies that g^r_d is not very ample, a contradiction. Hence $g^{r-1}_{d_{r-1}}$ is compounded. By [2, Lemma 5.3] we conclude that

$$g^{r-1}_{d_{r-1}} = (r-1)g^1_t$$

for a pencil g^1_t on X. Clearly $t \geq d_1$ and, since $(r-1)d_1 \geq d_{r-1} = (r-1)t$ by Lemma 3.1 (c), we must have $t = d_1$.

(2): In order to abbreviate, we write $m := \chi(d,r)$. Then $d - 1 = m(r-1) + p$ with $1 \leq p \leq r - 1$. Since $d \geq 3r - 1$ we have $m \geq 3$. Consequently, $(m-3)(r-1) + 2p \geq 2$ which implies that

$$\frac{m-2}{2}\left((m-3)(r-1) + 2p\right) \geq m - 2.$$

But one easily computes that

$$\begin{aligned} \pi(d,r) - (2d + 1 - 3r) &= (m-2)\left(d - 1 - \frac{m+3}{2}(r-1)\right) \\ &= \frac{m-2}{2}\left((m-3)(r-1) + 2p\right) \geq m - 2. \end{aligned}$$

This implies that $g > \pi(d,r) - m + 2 \geq 2d + 1 - 3r$, which is (2). □

Example 4.11. ($d = 10, r = 3$). A smooth curve X of type $(5,5)$ on a smooth quadric Q in $\mathbb{P}^3$ has genus $g = 16 = \pi(10,3)$ and gonality $d_1 = 5$ (see [11]). A smooth curve of type $(4,6)$ on Q has genus $g = 15$ and gonality $d_1 = 4$ (see [11]). By Proposition 4.10, $d_2 = 9 = \deg X - 1$ in the first case and $d_2 = 2d_1 = 8$ in the second case. (Note, however, that in the latter case X is an extremal curve of degree 14 in $\mathbb{P}^5$.) □

Example 4.11 can be generalized as follows:

Example 4.12. Let X be a smooth curve of type (a,b) with $a \leq b$ on a smooth quadric Q in $\mathbb{P}^3$. Then X has degree $d = a + b$, genus $g = (a-1)(b-1)$ and gonality $d_1 = a$ and, according to [5, Example 4.9], we have $d_2 = 2a$ if $a < b$ and $d_2 = d - 1$ if $a = b$. In the latter

case we have $g = (a-1)^2$ and so condition (2) in Lemma 4.8 just reads $(a-3)^2 > 0$. Hence for $a \geq 4$ Lemma 4.8 implies that a smooth curve of type (a, a) on Q does not satisfy all slope inequalities. Note that such a curve is a smooth complete intersection of Q with a surface of degree $a \geq 4$ and it is an extremal space curve of degree $2a$. □

By Proposition 2.4 the curve X in Proposition 4.10 is a smooth curve of degree d on a surface of degree $r-1$ in $\mathbb{P}^r$. In this case it is not difficult to compute d_1 (see [11]).

Theorem 4.13. *Let X be an extremal curve of degree $d \geq 3r-1$ in $\mathbb{P}^r$. Then $d_{r-1} = d-1$ and X does not satisfy all slope inequalities.*

Proof. By Lemma 4.8 and Proposition 4.10 it is enough to show that $(r-1)d_1 \geq d-1$ for an extremal curve of degree d in $\mathbb{P}^r$.

This is true for $r = 2$. For $r \geq 3$ it is known (combine [3, Section III, Theorem 2.5] and [11]) that

$$d_1 = \chi(d, r) + 1, \quad \text{if } r-1 \text{ does not divide } d-1,$$

and

$$\chi(d, r) + 1 \leq d_1 \leq \chi(d, r) + 2, \quad \text{if } r-1 \text{ divides } d-1,$$

unless $r = 5$, in which case X can also be isomorphic to a smooth plane curve of degree $\frac{d}{2}$, which has gonality $d_1 = \frac{d}{2} - 1$. This implies the assertion. □

Recall that by Proposition 2.1 the genera of the curves of Theorem 4.13 cover a big subset of $\mathbb{N}$.

The following example shows that a violation of the slope inequalities for members of the gonality sequence less than g is not restricted to smooth plane curves.

Example 4.14. *A smooth curve X of type $(5, 5)$ on a smooth quadric Q in $\mathbb{P}^3$ is an extremal space curve of degree 10 whose gonality sequence is for $r \leq g - d_1 - 1 = 10$:*

r	1	2	3	4	5	6	7	8	9	10
d_r	5	9	10	14	15	18	19	20	23	24

In particular $\frac{d_3}{3} < \frac{d_4}{4}$.

Proof. The first 3 members of the gonality sequence are clear by example 4.11. According to [2, Lemma 5.1] the linear series $g^3_{10} + g^1_5$ is a g^n_{15} with $n \geq 5$, and $n > 5$ is impossible by [7, Corollary 2.4.3]. By Serre duality it suffices to show that $d_4 = 14$.

Assume that X admits a g^4_{13}. Since $d_1 = 5$, we have $\dim(g^4_{13} - g^3_{10}) \leq 0$. Then [3, III Exercise B-6] implies that

$$\dim(g^4_{13} + g^3_{10}) \geq 2 \cdot 4 + 3 - 1 = 10.$$

However a linear series g^{10}_{23} is Serre-dual to a g^2_7 which contradicts $d_2 = 9$. □

By Section 2, the next example is not an extremal curve of degree $d \geq 3r - 1$ in $\mathbb{P}^r$.

Example 4.15. *A smooth complete intersection X of a cubic and a quartic surface in $\mathbb{P}^3$ has degree 12, genus 19, Clifford index $\gamma = 6 = d_1 - 2$ and does not satisfy all slope inequalities.*

Proof. By [4] we have $d_1 = 12 - \ell = \gamma + 2$ if X has an ℓ-secant line, but no $(\ell + 1)$-secant line.

According to [12, Lemma 2], X has a 4-secant line. It has no 5-secant line, since such a line would lie on both the cubic and the quartic surface intersecting in X and would thus be contained in X. Hence $d_1 = 8, \gamma = 6$ and X has genus 19 ([3, III, Ecercise C-1]).

Concerning the statement on the slope inequalities, according to Lemma 4.8 it is enough to show that $d_2 = 11$. So assume that X has a g^2_{10} (computing γ). Then, by [2, Lemma 5.1] we have $\dim(g^3_{12}+g^2_{10}) \geq 7$. If equality holds, we get the contradiction $d_2 = 2d_1 = 16$ in the same way as in the last part of the proof of Proposition 4.10 (1). So $g^3_{12} + g^2_{10}$ gives rise to a g^8_{22} on X, whose dual is a g^4_{14} computing γ. But according to [7, Corollary 3.2.5] there is no g^r_d on X computing γ such that $12 < d < g = 19$. Hence $d_2 = 10$ is impossible. □

In the Examples 3.3, 4.7 and 4.15 one can compute the Clifford index γ by a linear series which is not a pencil. The following proposition is a consequence of Lemma 4.8 for a curve with such a property.

Proposition 4.16. *Assume that $g \geq 2\gamma+r+2$ and $d_r = \gamma+2r$ for some $r \geq 2$. If X satisfies all slope inequalities, then for all $s = 1, \ldots, r$,*

$$d_s = \gamma + 2s.$$

Proof. By our assumption on g we have $g + s - 1 > \gamma + 2s$ for all $s = 1, \ldots, r$. So $\min\{\gamma + 2s, g + s - 1\} = \gamma + 2s$. According to Lemma 3.2 (d) we thus have $d_s \geq \gamma + 2s = d_r - 2(r - s)$ for $s = 1, \ldots, r$.

Hence, if X satisfies all slope inequalities, then Lemma 4.8 implies that $d_{r-1} = d_r - 2$. Repeating this argument gives the assertion. □

The curves fulfilling the hypotheses of Proposition 4.16 are good candidates for curves not satisfying all slope inequalities. We will demonstrate this in the next section.

5. Curves on a general K3-surface

We extended Corollary 4.6 to Theorem 4.13. Likewise, the curves of Example 4.15 generalize to a bigger class of curves not satifying all slope inequalities. We consider the following situation.

Let S be a general K3-surface of degree $2r-2$ in $\mathbb{P}^r$. Then Pic$S$ is generated by the class of a hyperplane section H and $H^2 = \deg S = 2r-2$. Let X be a smooth irreducible curves in S. Then X is contained in a linear series $|nH|$ for some positive integer n. The curve X is $\frac{1}{n}$-canonical (i.e. $K_X = \mathcal{O}_X(n)$), of genus $g = \frac{1}{2}X^2 + 1 = n^2(r-1)+1$ and degree $d = X \cdot H = 2n(r-1)$ in $\mathbb{P}^r$.

If $n \geq 2$, then by [7, Example 3.2.6] the linear system $|H|_X|$ computes the Clifford index $\gamma = d - 2r = 2(n-1)(r-1) - 2$ of X. For $n = r = 3$ we obtain a special case of Example 4.15.

Lemma 5.1. *In the situation just described, assume that $n \geq 4$. Let D be a base point free and effective divisor on X of degree $\delta < d$. Then D is contained in a hyperplane section of X (more precisely, $h^0(H|_X - D) = 1$ if $D \neq 0$).*

Proof. We apply Reider's method [15]: according to [15, Proposition 2.10, Remark 2.11, 1) and Corollary 1.40], for any base point free and effective divisor D on X of degree $\delta < \frac{1}{4}X^2$ there is an effective non-trivial divisor E_1 on S such that

$$E_1^2 < (X - E_1) \cdot E_1 \leq \delta, \quad \text{and} \quad E_1|_X - D \geq 0.$$

We apply this to our divisor D of the statement of the lemma. We have

$$\delta < d = 2n(r-1) \leq \frac{1}{2}n^2(r-1) = \frac{1}{4}X^2$$

for $n \geq 4$. So by Reider's method, there is a divisor E_1 on S with the indicated properties. Since Pic$S = \mathbb{Z}H$, we have $E_1 \in |\lambda H|$ for some positive integer λ. Then $E_1^2 < (X - E_1) \cdot E_1$ implies that $\lambda < \frac{n}{2}$, and from $(X - E_1) \cdot E_1 \leq \delta < d$ we obtain

$$\lambda^2 - n\lambda + n \geq 1.$$

Since λ in an integer, this is equivalent to

$$\left(\lambda - \frac{n}{2}\right)^2 \geq \left(\frac{n-2}{2}\right)^2.$$

But $\lambda - \frac{n}{2} < 0$ and $\frac{n-2}{2} > 0$, so this implies that $\lambda - \frac{n}{2} \leq -\frac{n-2}{2}$, i.e. $\lambda \leq 1$. Consequently, $|E_1| = |H|$, and so $E_1|X - D \geq 0$ shows that $h^0(H_X - D) \geq 1$. To prove equality here, observe that $\delta \geq d_1 \geq \gamma + 2$ if $D \neq 0$. Then

$$\begin{aligned} \deg(H|_X - D) = d - \delta &\leq d - (\gamma+2) = (\gamma + 2r) - (\gamma + 2) \\ &= 2(r-1) = \frac{\gamma+2}{n-1} < \gamma + 2 \leq d_1. \end{aligned}$$

This implies that $h^0(H|_X - D) = 1$. $\square$

Corollary 5.2. *In the situation of above let $n \geq 4$. Then $|H|_X|$ is the only g^r_d on X.*

Proof. Let $|L|$ be a g^r_d on X different from $|H|_X|$. Since $|L|$ computes γ, it is base point free and simple ([7]). Hence for a general divisor E in $|L|$ there is a point $p \in X$ in the support of E such that the divisor $E - p$ is base point free. So Lemma 5.1 implies that $E - p$ is contained in $|H|_X|$. But by the General Position Theorem ([2, Theorem 4.1]), the greatest common divisor of E and any divisor in $|H|_X|$ has degree at most r. Since $r < d - 1$, we get a contradiction. □

By the way, by Theorem 2.3 also all curves of Theorem 4.13 have merely one g^r_d.

Combining Lemma 4.8 (or Proposition 4.16) with Lemma 5.1 we obtain our second main result.

Theorem 5.3. *Let X be a curve contained in a general K3-surface of degree $2r - 2$ in $\mathbb{P}^r$. If $d := \deg X \geq 8(r - 1)$, then $d_{r-1} = d - 1$ and X does not satisfy all slope inequalities.*

Proof. A $g^{r-1}_{d_{r-1}}$ on X is contained in the $g^r_d = |H|_X|$, by Lemma 5.1. If $d_{r-1} \leq d - 2$, this contradicts the very ampleness of the g^r_d. Hence $d_{r-1} = d - 1$, and we can apply Lemma 4.8. □

According to Dirichlet's prime number theorem, the function

$$f(r) := n^2(r - 1) + 1$$

represents, for every integer $n \neq 0$, infinitely many prime numbers. Hence, for fixed $n \geq 4$, Theorem 5.3 produces an infinite number of curves of prime genus. By Section 2, this cannot be achieved by virtue of Theorem 4.13. However, this does not yet answer the following question:

Question 5.4: Is there a number $g_0 \geq 9$ such that for every integer $g \geq g_0$ there is a curve of genus g not satisfying all slope inequalities?

6. Space curves

The curves of Examples 4.7, 4.12 (with $a = b$), 4.14 and 4.15 are complete intersections of two surfaces in $\mathbb{P}^3$. More generally, based on [5], we have the following theorem.

Theorem 6.1. *Let X be a smooth complete intersection of two surfaces of degree p and s with $p \geq s \geq 2$ in $\mathbb{P}^3$. Then X is a space curve of degree $d := ps$ and genus $g = \frac{1}{2}ps(p+s-4)+1$, and we have $d_2 = d-1$. If $p \geq 4$, then X does not satisfy all slope inequalities.*

Proof. Since $\mathcal{O}_X(p + s - 4)$ is the canonical bundle of X, we have $2g - 2 = d(p + s - 4)$ ([3, III, Exercise C-1]). In [5, Example 4.6] it is shown that $d_2 = d - 1$. The condition (2) in Lemma 4.8 is equivalent to

$$ps(8 - p - s) < 18,$$

and this is fulfilled if $p \geq 4$ (i.e if $(s,p) \neq (2,2),(2,3),(3,3)$). So Lemma 4.8 gives the last assertion of the theorem. $\square$

By [4] the curve X in Theorem 6.1 has gonality $d_1 = d - \ell$ if X has an ℓ-secant line, but no $(\ell+1)$-secant line. By [13], $\ell \leq s$ or $\ell = p$. Let $p \geq 4$. Then $\gamma = d_1 - 2$ ([4]) and $\ell \geq 4$ ([12, Lemma 2]). In fact, according to [9, Corollary 5.2], if X is general in its linear series on a smooth surface S of degree s and if S does not contain a line, then $s \geq 4$ and $\ell = 4$. So the linear series of plane sections computes $\gamma = d - 6$ for a "general" complete intersection X of two surfaces of degrees ≥ 4 in $\mathbb{P}^3$.

The curve X in Theorem 6.1 is a special case of a Halphen curve which is defined as follows.

Definition 6.2. A smooth and irreducible space curves X of degree d is called a *Halphen curve* if there is an integer $s \geq 2$ such that $d > s(s-1)$ and if X is of maximal genus among those smooth and irreducible space curves of degree d which do not lie on a surface of degree $< s$.

If $d = ks - \epsilon$ with $0 \leq \epsilon < s$, then X has genus

$$g = G(d,s) := \frac{d^2}{2s} + \frac{d(s-4)}{2} + 1 - \frac{\epsilon}{2}(s - 1 - \epsilon + \frac{\epsilon}{s}),$$

which is Halphen's bound ([5, Remark 4.3], note the typo there). $\square$

Theorem 6.3. *Let X be a Halphen curve of degree d. Then $d_2 = d-1$ and if $d \geq 11$, then X does not satisfy all slope inequalities.*

Proof. According to Theorem 6.1 and [5, Theorem 4.7] we have $d_2 = d-1$. In order to apply Lemma 4.8 we have to check that $g-1 > 2d-9$.

The function

$$f(\epsilon) := \frac{\epsilon}{2}(s-1-\epsilon+\frac{\epsilon}{s}) = \frac{s-1}{2s}\epsilon(s-\epsilon)$$

is maximal for $\epsilon = \frac{s}{2}$. So $f(\epsilon) \leq \frac{1}{8}s(s-1)$ for $0 \leq \epsilon < s$. Hence $g - 1 > 2d - 9$ is satisfied if

$$\frac{d^2}{2s} + \frac{d(s-4)}{2} - f(\epsilon) \geq \frac{d^2}{2s} + \frac{d(s-4)}{2} - \frac{s(s-1)}{8} > 2d - 9,$$

i.e. if

$$d^2 + s(s-8)d + 18s > \frac{1}{4}s^2(s-1).$$

Recall that $d > s(s-1)$. So this inequality is easily seen to be valid for $s \geq 4$. For $s = 2$ it is valid for $d \geq 8$ and for $s = 3$ for $d \geq 11$. $\square$

Note that the curves of Theorems 4.13, 5.3, 6.1 and 6.3 are projectively normal (= arithmetically Cohen-Macaulay) under their embedding considered. One may ask the following question:

Question 6.4. Does a smooth, irreducible and projectively normal curve of degree d in $\mathbb{P}^r$ have $d_{r-1} = d - 1$?

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